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Bohr radius for the punctured disk

Yusuf Abu Muhanna[*1], **Rosihan M. Ali**[**2], and **Zhen Chuan Ng**[***2]

[1] Department of Mathematics, American University of Sharjah, Sharjah, Box 26666, United Arab Emirates
[2] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia

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This paper investigates the Bohr phenomenon for the class of analytic functions from the unit disk into the punctured unit disk. The Bohr radius is shown to be $1/3$.

1 Introduction

Let $U := \{z : |z| < 1\}$ denote the unit disk and $H(U)$ be the class of analytic functions defined in U. In 1914, Bohr [14] proved that for every analytic self-map $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of the unit disk U, then

$$\sum_{n=0}^{\infty} |a_n z^n| \leq 1 \tag{1.1}$$

in the disk $0 \leq |z| \leq 1/6$. The value $1/6$ was further improved independently by Riesz, Schur and Wiener to $1/3$, which is the largest $|z|$ could be for inequality (1.1) to hold. The number $1/3$ is now known as the Bohr radius for the class of all analytic self-maps of the unit disk U. Other proofs can also be found in [24], [27], [28].

Greater interest is given to the Bohr theorem after Dixon [19] used it to construct a non-unital Banach algebra, which is not an operator algebra, but yet satisfy the non-unital von Neumann's inequality. Generalizations of Bohr theorem were studied by various authors: Aizenberg [2], [3] considered the domain of unit ball and unit hypercone; Popescu, Paulsen and Singh [23], [25], [26] established the operator-theoretic Bohr radius; Aizenberg, Aytuna and Djakov [5], [7] described the Bohr property of bases for holomorphic functions; Bénéteau, Dahlner and Khavinson [10] studied the Bohr phenomenon for functions in Hardy spaces; and Ali, Abdulhadi and Ng [6] extended the concept of the Bohr radius to the class of starlike logharmonic mappings. The interconnectivity between Banach theory and Bohr theorem was investigated in [11], [15], [16].

The generalization of Bohr theorem to higher dimensions was pioneered by Dineen and Timoney [18] in the setting of the unit polydisk, which led to a partial solution of the problem. Several years later, Boas and Khavinson [13] provided the estimate for the n-dimensional Bohr radius. Recently, Defant et al. [17] obtained the optimal asymptotic estimate for this radius by using the fact that the Bohnenblust–Hille inequality is indeed hypercontractive. The exact asymptotic behaviour of the radius was obtained by Bayart, Pellegrino and Seoane-Sepúlveda [8].

Bohr inequality (1.1) can also be written as

$$d\left(\sum_{n=0}^{\infty} \left|a_n z^n\right|, |a_0|\right) = \sum_{n=1}^{\infty} \left|a_n z^n\right| \leq d(f(0), \partial U),$$

where d is the Euclidean distance, and ∂U is the boundary of U. This form makes evident the notion of the Bohr phenomenon for analytic functions mapping the unit disk into a given domain. Let $S(\Omega)$ be the class consisting

* e-mail: ymuhanna@aus.edu
** Corresponding author: e-mail: rosihan@usm.my
*** e-mail: zc_ng2004@yahoo.com

of all analytic (or harmonic) functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ from U into a domain Ω. The Bohr radius for Ω is the largest number $r_\Omega \in (0, 1)$ satisfying

$$d\left(\sum_{n=0}^{\infty} \left|a_n z^n\right|, |a_0|\right) = \sum_{n=1}^{\infty} \left|a_n z^n\right| \leq d(f(0), \partial\Omega)$$

for all $f \in S(\Omega)$ and $|z| < r_\Omega$.

If Ω is convex, Aizenberg [4] proved that the sharp Bohr radius is $r_\Omega = 1/3$. This result includes the classical case $\Omega = U$. When Ω is any proper simply connected domain, Abu-Muhanna [1] showed that the Bohr radius is $3 - 2\sqrt{2} \cong 0.17157$. In two recent papers [21], [22], the Bohr inequality was investigated when Ω is the domain exterior to a compact convex set, and when Ω is a concave-wedge domain.

The aim of this paper is to investigate the Bohr phenomenon for the class of analytic functions mapping U into the punctured unit disk. The sharp Bohr radius of $1/3$ for the punctured unit disk is obtained in Theorem 2.6. A similar Bohr-type inequality is also obtained in Theorem 2.10. Additionally, Theorem 2.11 yields the Bohr inequality involving the hyperbolic metric.

2 Analytic functions mapping into the punctured unit disk

Denote by $U_0 = U\backslash\{0\}$ the unit disk punctured at the origin, and $\overline{U}_r$ the disk $\{z : |z| \leq r\}$. Further, denote by $\mathcal{H}$ the class of all analytic self-maps of U, and

$$\mathcal{H}_0 := \{f \in \mathcal{H} : f(U) \subseteq U_0\}.$$

Since $f(z) \neq 0$ in U whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}_0$, evidently $|a_0| > 0$.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U)$, the *majorant function* [12] is given by

$$\mathcal{M}f(z) := \sum_{n=0}^{\infty} |a_n| z^n.$$

Thus the classical Bohr theorem (1.1) takes the form

$$\mathcal{M}f(|z|) \leq 1$$

for $|z| \leq 1/3$ and $f \in \mathcal{H}$. Since $|\mathcal{M}f(z)| \leq \mathcal{M}f(|z|)$, Bohr theorem can be written as

$$\mathcal{M}f(\overline{U}_{1/3}) \subseteq U$$

for $f \in \mathcal{H}$, or in distance form,

$$d\left(\mathcal{M}f(|z|), |f(0)|\right) \leq d(f(0), \partial U)$$

for $|z| \leq 1/3$ and $f \in \mathcal{H}$.

The following theorem shows that the Bohr radius $1/3$ also holds for the subclass $\mathcal{H}_0$ of $\mathcal{H}$.

Theorem 2.1 *If $f \in \mathcal{H}_0$, then*

$$\mathcal{M}f(\overline{U}_{1/3}) \subseteq U, \tag{2.1}$$

or equivalently,

$$d\left(\mathcal{M}f(|z|), |f(0)|\right) \leq d(f(0), \partial U) \tag{2.2}$$

for $|z| \leq 1/3$. The radius $1/3$ is best.

Proof. Since $f \in \mathcal{H}_0 \subset \mathcal{H}$, the inclusion (2.1) follows immediately from the classical Bohr theorem. To show the value $1/3$ is best, consider the function

$$\begin{aligned} f_t(z) &= \exp\left(-t\frac{1+z}{1-z}\right) \\ &= \frac{1}{e^t} + \frac{1}{e^t}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right]z^n, \quad t>0. \end{aligned} \tag{2.3}$$

Note that

$$\left|\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right| \geq -\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1},$$

and so

$$\begin{aligned} \mathcal{M}f_t(|z|) &\geq \frac{1}{e^t} - \frac{1}{e^t}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right]|z|^n \\ &= \frac{2}{e^t} - f_t(|z|). \end{aligned} \tag{2.4}$$

Let $a_0 = f_t(0)$. Since $t = -\log a_0 = -\log|a_0|$, f_t can be written as

$$\begin{aligned} f_t(z) = \exp\left(\log|a_0|\frac{1+z}{1-z}\right) &= |a_0|\exp\left(\log|a_0|\frac{2z}{1-z}\right) \\ &= |a_0||a_0|^{\frac{2z}{1-z}}. \end{aligned} \tag{2.5}$$

Hence, by letting $|z| = r$, (2.4) and (2.5) imply

$$\mathcal{M}f_t(|z|) \geq 2|a_0| - f_t(|z|) = |a_0|(2 - |a_0|^{\frac{2r}{1-r}}) > 1$$

as $a_0 \to 1$ and $r > 1/3$. Indeed, for each $r_0 > 1/3$, there exists an $\epsilon_0 > 0$ satisfying

$$1 < \frac{-\log(1-\epsilon_0)}{\log(1+\epsilon_0)} < \frac{2r_0}{1-r_0}.$$

Equivalently,

$$1 - \epsilon_0 > (1+\epsilon_0)^{-\frac{2r}{1-r}}.$$

With $|a_0| = 1/(1+\epsilon_0)$, then

$$|a_0|\left(2 - |a_0|^{\frac{2r}{1-r}}\right) > 1,$$

which gives $\mathcal{M}f_t(r_0) > 1$. Also note that

$$|a_0|\left(2 - |a_0|^{\frac{2r}{1-r}}\right) \leq 2|a_0| - |a_0|^2 \leq 1$$

for $|a_0| < 1$ and $0 \leq r \leq 1/3$. Hence the radius $1/3$ is best. □

Since $f(U) \subseteq U_0$, the Bohr theorem for the class $\mathcal{H}_0$ suggests replacing the domain U in both (2.1) and (2.2) by U_0. To this end, we first examine the case for functions $f_t \in \mathcal{H}_0$ given by (2.3). For such functions f_t, Koepf and Schmersau [20, p. 248] obtained the estimate

$$\left|\frac{1}{e^t}\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right| < \sqrt{\frac{2t}{n}}, \quad t \in (0, 2n), \quad n > 0. \tag{2.6}$$

Also, it would soon become evident that the number

$$\alpha_0 := \frac{1}{3e} - \frac{1}{9\sqrt{6}} \approx 0.07727$$

plays a prominent role in the sequel.

Lemma 2.2 *Let f_t be given by* (2.3) *with* $0 < t \le 1$. *Then* $\mathcal{M} f_t(\overline{U}_{1/3}) \subseteq U_0$ *and*

$$\left| \mathcal{M} f_t(z) - \frac{1}{e^t} \right| < \frac{1}{e^t} - \alpha_0, \quad z \in \overline{U}_{1/3}.$$

In particular,

$$\mathcal{M} f_t(|z|) - \frac{1}{e^t} < \frac{1}{e^t} - \alpha_0, \quad |z| \le 1/3.$$

Proof. Write

$$\begin{aligned} f_t(z) &= \exp\left(-t\frac{1+z}{1-z}\right) = \exp\left(-t - 2t\sum_{n=1}^{\infty} z^n\right) \\ &= \frac{1}{e^t} + \sum_{m=1}^{\infty} \frac{1}{e^t m!}\left(-2t\sum_{n=1}^{\infty} z^n\right)^m \\ &= \frac{1}{e^t} - \frac{2t}{e^t}z - \frac{2t(1-t)}{e^t}z^2 + a_3 z^3 + \cdots . \end{aligned}$$

Thus for $|z| \le 1/3$, (2.6) gives

$$\begin{aligned} \left| \mathcal{M} f_t(z) - \frac{1}{e^t} \right| &< \frac{2t}{3e^t} + \frac{2t(1-t)}{9e^t} + \sum_{n=3}^{\infty} \frac{\sqrt{2t}}{3^n\sqrt{n}} \\ &< \frac{2t}{3e^t} + \frac{2t(1-t)}{9e^t} + \sqrt{\frac{2t}{3}} \sum_{n=3}^{\infty} \frac{1}{3^n} \\ &= \frac{2t}{3e^t} + \frac{2t(1-t)}{9e^t} + \frac{1}{9}\sqrt{\frac{t}{6}} \\ &= \frac{1}{e^t} - y_1(t) < \frac{1}{e^t} - \alpha_0, \end{aligned}$$

where

$$y_1(t) := \frac{1}{e^t} - \frac{2t}{3e^t} - \frac{2t(1-t)}{9e^t} - \frac{1}{9}\sqrt{\frac{t}{6}}$$

is strictly decreasing in $[0, 1]$. Thus $y_1(t) > \alpha_0$ in $[0, 1]$. It follows that $|\mathcal{M} f_t(z)| > 0$, which along with Theorem 2.1 give $\mathcal{M} f_t(\overline{U}_{1/3}) \subseteq U_0$. □

Remark 2.3 Equation y_1 in Lemma 2.2 has a root at $t_0 \approx 1.35299$, and indeed y_1 is strictly decreasing in $[0, t_0]$. We shall however be only interested in the interval $t \in (0, 1]$.

Lemma 2.4 *Let $f_{a,N} \in \mathcal{H}_0$ be of the form*

$$f_{a,N}(z) = \exp\left(-\sum_{k=1}^{N} t_k a \frac{1 + x_k z}{1 - x_k z}\right), \quad z \in U, \tag{2.7}$$

with $0 < a \le 1$, $|x_k| = 1$ *for each* k, *and* $t_k > 0$ *satisfies* $\sum_{k=1}^{N} t_k = 1$. *Then* $\mathcal{M} f_{a,N}(\overline{U}_{1/3}) \subseteq U_0$ *and*

$$\left| \mathcal{M} f_{a,N}(z) - \frac{1}{e^a} \right| < \frac{1}{e^a} - \alpha_0, \quad z \in \overline{U}_{1/3}.$$

Proof. Since $f_{a,N}$ is analytic in U, it can be expressed in its Taylor series

$$\begin{aligned}
\exp\left(-\sum_{k=1}^{N} t_k a \frac{1+x_k z}{1-x_k z}\right) &= \exp\left(-a - 2a \sum_{n=1}^{\infty}\left(\sum_{k=1}^{N} t_k x_k^n\right) z^n\right) \\
&= \frac{1}{e^a} \exp\left(-2a \sum_{n=1}^{\infty}\left(\sum_{k=1}^{N} t_k x_k^n\right) z^n\right) \\
&= \frac{1}{e^a} + \frac{1}{e^a} \sum_{m=1}^{\infty} \frac{(-2a)^m}{m!} \left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N} t_k x_k^n\right) z^n\right)^m \\
&= \frac{1}{e^a} + \frac{1}{e^a} \sum_{m=1}^{\infty} \frac{(-2a)^m}{m!} \sum_{n=m}^{\infty} d_n z^n \\
&= \frac{1}{e^a} + \frac{1}{e^a} \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{(-2a)^m}{m!} d_n z^n \\
&= \frac{1}{e^a} + \frac{1}{e^a} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(-2a)^m}{m!} d_n z^n,
\end{aligned}$$

where

$$d_n = \sum_{s_1+\cdots+s_m=n} \left(\sum_{k=1}^{N} t_k x_k^{s_1}\right) \cdots \left(\sum_{k=1}^{N} t_k x_k^{s_m}\right)$$

and the outer sum is taken over all m-tuples $(s_1, \ldots, s_m)$ of postive integers satisfying $s_1 + \cdots + s_m = n$. Note that

$$|d_n| \leq \sum_{s_1+\cdots+s_m=n} \left(\sum_{k=1}^{N} t_k\right) \cdots \left(\sum_{k=1}^{N} t_k\right) = \sum_{s_1+\cdots+s_m=n} 1 = \binom{n-1}{m-1}.$$

Next let

$$f_a(z) = \exp\left(-a\frac{1+z}{1-z}\right) = \frac{1}{e^a} + \frac{1}{e^a} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(-2a)^m}{m!} \binom{n-1}{m-1} z^n.$$

Thus for $|z| \leq 1/3$,

$$\begin{aligned}
\left|\mathcal{M} f_{a,N}(z) - \frac{1}{e^a}\right| &\leq \frac{1}{e^a} \sum_{n=1}^{\infty} \left|\sum_{m=1}^{n} \frac{(-2a)^m}{m!}\right| |d_n| |z|^n \\
&\leq \mathcal{M} f_a(|z|) - \frac{1}{e^a} < \frac{1}{e^a} - \alpha_0,
\end{aligned}$$

where the last inequality follows from Lemma 2.2. Hence $\left|\mathcal{M} f_{a,N}(z)\right| > 0$ on $\overline{U}_{1/3}$, which together with Theorem 2.1 yields $\mathcal{M} f_{a,N}(\overline{U}_{1/3}) \subseteq U_0$. □

Theorem 2.5 *Let $f \in \mathcal{H}_0$ with $1/e \leq |f(0)| < 1$. Then $\mathcal{M} f(\overline{U}_{1/3}) \subseteq U_0$ and*

$$\left|\mathcal{M} f(z) - \mathcal{M} f(0)\right| < \mathcal{M} f(0), \quad z \in \overline{U}_{1/3}.$$

In particular,

$$\mathcal{M} f(|z|) - |f(0)| < |f(0)|, \quad |z| \leq 1/3.$$

Proof. It suffices to consider the case $f(0) > 0$. Since $0 < |f(z)| < 1$, then $-\mathrm{Re}\, \log f(z) > 0$ in U. Thus,

$$\log f(z) = \log f(0) \int_{|x|=1} \frac{1+xz}{1-xz}\, d\mu(x),$$

or

$$f(z) = \exp\left(-\int_{|x|=1} a\frac{1+xz}{1-xz}\, d\mu(x)\right)$$

for some probability measure μ on ∂U, and $0 < a = -\log f(0) \le 1$. If f has the form (2.7), then the results evidently follow from Lemma 2.4.

Consider the compact disk $\overline{U}_\rho$ with $1/3 < \rho < 1$. If f does not has the form (2.7), then there exists a sequence of functions $\{g_n\}$ of the form (2.7) satisfying $g_n(0) = f(0)$ for each n, and g_n converges uniformly to f on $\overline{U}_\rho$. Thus for a given $\epsilon > 0$, there exists a positive integer N such that

$$|g_n(z) - f(z)| < \frac{\epsilon}{M} \quad \text{for all } \ z \in \overline{U}_\rho,$$

and $n > N$, where $M = \max_{z\in\overline{U}_{1/3}}\{|z|/(\rho - |z|)\}$. The Cauchy Integral formula yields

$$\begin{aligned}\left|g_n^{(k)}(0) - f^{(k)}(0)\right| &= \left|\frac{k!}{2\pi i}\oint_{\partial\overline{U_\rho}} \frac{g_n(\zeta) - f(\zeta)}{\zeta^{k+1}}\, d\zeta\right| \\ &\le \frac{k!}{2\pi}\int_0^{2\pi} \frac{|g_n(\zeta(t)) - f(\zeta(t))|}{\rho^k}\, dt < \frac{\epsilon k!}{M\rho^k}.\end{aligned}$$

Hence, for all $|z| \le 1/3$ and $n > N$,

$$\begin{aligned}|\mathcal{M}g_n(z) - \mathcal{M}f(z)| &\le |\mathcal{M}(g_n - f)(|z|)| \\ &= \sum_{k=1}^{\infty}\left|\frac{g_n^{(k)}(0) - f^{(k)}(0)}{k!}\right||z|^k \\ &< \frac{\epsilon}{M}\sum_{k=1}^{\infty}\left(\frac{|z|}{\rho}\right)^k = \frac{\epsilon|z|}{M(\rho - |z|)} \le \epsilon,\end{aligned}$$

implying $\mathcal{M}g_n \to \mathcal{M}f$ uniformly on $\overline{U}_{1/3}$.

Now, for any $\epsilon > 0$, there exists a corresponding positive integer N such that

$$\sup_{z\in\overline{U}_{1/3}} |\mathcal{M}g_n(z) - \mathcal{M}f(z)| < \epsilon \quad \text{for all } \ n > N.$$

Lemma 2.4 and the inequality above imply

$$\begin{aligned}\sup_{z\in\overline{U}_{1/3}} |\mathcal{M}f(z) - f(0)| &\le \sup_{z\in\overline{U}_{1/3}} |\mathcal{M}g_n(z) - f(0)| + \sup_{z\in\overline{U}_{1/3}} |\mathcal{M}g_n(z) - \mathcal{M}f(z)| \\ &< f(0) - \alpha_0 + \epsilon.\end{aligned}$$

Hence $|\mathcal{M}f(z) - f(0)| \le f(0) - \alpha_0 < f(0)$ for all $z \in \overline{U}_{1/3}$, and so $\left|\mathcal{M}f(z)\right| > 0$ on $\overline{U}_{1/3}$. Further Theorem 2.1 gives $\mathcal{M}f(\overline{U}_{1/3}) \subseteq U_0$. □

The following result yields the Bohr radius for the class $\{f \in \mathcal{H}_0 : 1/e \le |f(0)| < 1\}$.

Theorem 2.6 *If $f \in \mathcal{H}_0$ with $1/e \le |f(0)| < 1$, then*

$$\mathcal{M}f(\overline{U}_{1/3}) \subseteq U_0 \tag{2.8}$$

and

$$d(\mathcal{M}f(|z|), |f(0)|) \le d(f(0), \partial U_0) \tag{2.9}$$

for $|z| \le 1/3$. The radius $1/3$ is best possible.

P r o o f . The inclusion (2.8) follows from Theorem 2.5. Now, assume that $r = |z| \le 1/3$. The inequality in Theorem 2.5 implies

$$d(\mathcal{M}f(r), |f(0)|) = \mathcal{M}f(r) - |f(0)| < |f(0)|.$$

On the other hand, since $f \in \mathcal{H}_0$, Theorem 2.1 gives $\mathcal{M}f(r) < 1$ and so

$$d(\mathcal{M}f(r), |f(0)|) = \mathcal{M}f(r) - |f(0)| < 1 - |f(0)|.$$

Then (2.9) follows from the two inequalities above since

$$d(f(0), \partial U_0) = \min\{|f(0)|, 1 - |f(0)|\}.$$

That the value $1/3$ is best follows from the proof of Theorem 2.1. □

Remark 2.7 Relations (2.1) and (2.2) in Theorem 2.1 are equivalent. However (2.8) and (2.9) in Theorem 2.6 are not since $d(f(0), \partial U_0) = |f(0)| \neq 1 - |f(0)|$ for $|f(0)| < 1/2$.

Next, we look at removing the constraint on $|f(0)|$ in Theorem 2.6. Denote by $[\cdot]$ the least integer function, that is, $[a]$ is the smallest integer greater than or equal to a.

Lemma 2.8 *Suppose $a > 0$, and*

$$f_{a,N}(z) = \exp\left(-\sum_{k=1}^{N} t_k a \frac{1 + x_k z}{1 - x_k z}\right) \in \mathcal{H}_0, \tag{2.10}$$

where $|x_k| = 1$ for each k, and $t_k > 0$ satisfies $\sum_{k=1}^{N} t_k = 1$. Then

$$(\mathcal{M}f_{a,N}(|z|))^{1/[a]} - \frac{1}{e^{a/[a]}} < \frac{1}{e^{a/[a]}} - \alpha_0, \quad |z| \leq 1/3.$$

P r o o f. If $a \in (0, 1]$, then $[a] = 1$ and the result follows from Lemma 2.4. Assume now that $a > 1$. It follows from the proof of Lemma 2.4 that

$$\left|\mathcal{M}f_{a,N}(z) - \frac{1}{e^a}\right| \leq \mathcal{M}f_a(|z|) - \frac{1}{e^a},$$

which gives

$$\left|\mathcal{M}f_{a,N}(z)\right| \leq \mathcal{M}f_a(|z|). \tag{2.11}$$

Since $\mathcal{M}(fg)(|z|) \leq \mathcal{M}(f)(|z|)\mathcal{M}(g)(|z|)$, Lemma 2.2 yields

$$\mathcal{M}f_a(|z|) = \mathcal{M}f_{a/[a]}^{[a]}(|z|) \leq \left(\mathcal{M}f_{a/[a]}(|z|)\right)^{[a]} < \left(\frac{2}{e^{a/[a]}} - \alpha_0\right)^{[a]} \tag{2.12}$$

for $|z| \leq 1/3$. Thus

$$(\mathcal{M}f_{a,N}(|z|))^{1/[a]} - \frac{1}{e^{a/[a]}} < \frac{1}{e^{a/[a]}} - \alpha_0.$$ □

Theorem 2.9 *Let $f \in \mathcal{H}_0$ and $a = -\log|f(0)|$. Then*

$$(\mathcal{M}f(|z|))^{1/[a]} - |f(0)|^{1/[a]} < |f(0)|^{1/[a]}, \quad |z| \leq 1/3.$$

P r o o f. It suffices to consider the case $f(0) > 0$. Let $a = -\log f(0)$. Then

$$f(z) = \exp\left(-\int_{|x|=1} a \frac{1 + xz}{1 - xz}\, d\mu(x)\right)$$

for some probability measure μ on ∂U. If f has the form (2.10), then the result follows from Lemma 2.8.

Consider the compact disk $\overline{U}_\rho$ with $1/3 < \rho < 1$. If f does not has the form (2.10), then there exists a sequence of functions $\{g_n\}$ of the form (2.10) satisfying $g_n(0) = f(0)$ for each n, and g_n converges uniformly to f on $\overline{U}_\rho$. Applying the same argument as in the proof of Theorem 2.5, it can be shown that $\mathcal{M}g_n$ converges to $\mathcal{M}f$ uniformly on $\overline{U}_{1/3}$.

Thus, for any $\epsilon > 0$, there exists a corresponding positive integer N such that for all $n \geq N$ and $z \in \overline{U}_{1/3}$,

$$|\mathcal{M}g_n(z) - \mathcal{M}f(z)| < \epsilon,$$

and thus

$$|\mathcal{M}f(z)| < |\mathcal{M}g_n(z)| + \epsilon.$$

Further (2.11) and (2.12) imply

$$|\mathcal{M}f(z)| < \left(2\left(f(0)\right)^{1/[a]} - \alpha_0\right)^{[a]} + \epsilon.$$

Since ϵ is arbitrary, it follows that

$$|\mathcal{M}f(z)| \le \left(2\left(f(0)\right)^{1/[a]} - \alpha_0\right)^{[a]} < \left(2\left(f(0)\right)^{1/[a]}\right)^{[a]},$$

and consequently

$$\left(\mathcal{M}f(|z|)\right)^{1/[a]} - \left(f(0)\right)^{1/[a]} < \left(f(0)\right)^{1/[a]}$$

for $|z| \le 1/3$. □

Theorem 2.10 *If $f \in \mathcal{H}_0$ and $a = -\log|f(0)|$, then*

$$d\left(\left(\mathcal{M}f(|z|)\right)^{1/[a]}, \left|f(0)\right|^{1/[a]}\right) \le d\left(\left(f(0)\right)^{1/[a]}, \partial U_0\right)$$

for $|z| \le 1/3$. The radius $1/3$ is best possible.

P r o o f . The proof is omitted as it can be argued likewise in proving (2.9) by applying Theorem 2.9.

To show the value $1/3$ is best, consider the function $f_t \in \mathcal{H}_0$ given by (2.3) with $1/2 \le f_{t/[t]}(0) < 1$. Then it suffices to show that

$$d\left(\left(\mathcal{M}f_t(|z|)\right)^{1/[t]}, \left(f_t(0)\right)^{1/[t]}\right) > d\left(\left(f_t(0)\right)^{1/[t]}, \partial U_0\right), \quad |z| > 1/3. \tag{2.13}$$

Since

$$d\left(\left(\mathcal{M}f_t(|z|)\right)^{1/[t]}, \left(f_t(0)\right)^{1/[t]}\right) = \left(\mathcal{M}f_t(|z|)\right)^{1/[t]} - \left(f_t(0)\right)^{1/[t]},$$

and

$$d\left(\left(f_t(0)\right)^{1/[t]}, \partial U_0\right) = 1 - \left(f_t(0)\right)^{1/[t]},$$

it follows that (2.13) can be reduced to

$$\left(\mathcal{M}f_t(|z|)\right)^{1/[t]} > 1 \quad \text{or} \quad \mathcal{M}f_t(|z|) > 1, \quad |z| > 1/3.$$

Indeed, the inequality holds as is shown in the proof of Theorem 2.1. □

We end this section by presenting a Bohr-type inequality in hyperbolic distance on U_0. The density of the metric [9] on U_0 is given by

$$\lambda_{U_0}(z) = \frac{1}{|z|\log(1/|z|)}.$$

If $d_{U_0}(a, b)$ denote the hyperbolic distance between a and b, then

$$d_{U_0}(a, b) = \int_a^b \frac{|dz|}{|z|\log(1/|z|)} = \log\left|\frac{\log 1/|b|}{\log 1/|a|}\right|.$$

Theorem 2.11 *Let $f \in \mathcal{H}_0$ with $1/e \le |f(0)| < 1$. Then*

$$d_{U_0}(\mathcal{M}f(|z|), |f(0)|) \le \log\frac{1 + 3|z|}{1 - 3|z|}$$

for $|z| < 1/3$. In particular,

(a) *when* $|z| < 1/9$,

$$d_{U_0}(\mathcal{M}f(|z|), |f(0)|) < \log 2 = \frac{2}{\lambda_{U_0}(\frac{1}{2})};$$

(b) *when* $|z| < (e-1)/3(1+e) \approx 0.15404$,

$$d_{U_0}(\mathcal{M}f(|z|), |f(0)|) < 1 = \frac{e}{\lambda_{U_0}(\frac{1}{e})};$$

(c) *when* $|z| < (1-|f(0)|)/3(1+|f(0)|)$,

$$d_{U_0}(\mathcal{M}f(|z|), |f(0)|) < \frac{1/|f(0)|}{\lambda_{U_0}(|f(0)|)}.$$

P r o o f . By Theorem 2.5, $\mathcal{M}f(U_{1/3}) \subseteq U_0$. Define a covering map $F : U \to U_0$ by

$$F(z) = \exp\left(\log(|f(0)|)\frac{1+z}{1-z}\right).$$

Also, the conformal map $\psi(z) = 3z$ sends $U_{1/3}$ onto U. Note that $F \circ \psi : U_{1/3} \to U_0$ is also a covering map. Thus by [9, Theorem 10.5],

$$\begin{aligned} d_{U_0}(|f(0)|, \mathcal{M}f(|z|)) &\leq d_{U_0}((F \circ \psi)(0), (F \circ \psi)(|z|)) \\ &= d_{U_{1/3}}(0, |z|) = d_U(\psi(0), \psi(|z|)) \\ &= d_U(0, 3|z|) = \log \frac{1+3|z|}{1-3|z|} \end{aligned}$$

in $U_{1/3}$. Part (a) and (b) are evident. For part (c), an upper bound for $|z|$ is obtained by solving the inequality

$$\log \frac{1+3|z|}{1-3|z|} < \frac{1/|f(0)|}{\lambda_{U_0}(|f(0)|)} = \log\left(\frac{1}{|f(0)|}\right). \qquad \square$$

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